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Capelli identities for symmetric pairs

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1 Introduction

Consider a see-saw pair of real reductive Lie groups in the real symplectic group $Sp_{2N}(\mathbf{R})$,

$$\begin{array}{ccc} G_0 & & M_0 \\ \cup & \times & \cup \\ K_0 & & H_0, \end{array}$$

where both (G_0, H_0) and (K_0, M_0) form dual pairs. The pair (G_0, H_0) is called a dual pair, if G_0 and H_0 are the commutants of each other in $Sp_{2N}(\mathbf{R})$. In addition, we assume that (G_0, K_0) is a symmetric pair of Hermitian type. Then there are three types of such see-saw pairs as in Table 1 [How89]. Note that (M_0, H_0) is also

表 1: see-saw pairs with G_0 Hermitian type

	$Sp_{2N}(\mathbf{R})$	G_0	K_0	M_0	H_0
Case **R**	$Sp_{2k(p+q)}(\mathbf{R})$	$Sp_{2k}(\mathbf{R})$	U_k	$U(p,q)$	$O(p,q)$
Case **C**	$Sp_{2(p+q)(r+s)}(\mathbf{R})$	$U(p,q)$	$U_p \times U_q$	$U(r,s) \times U(r,s)$	$U(r,s)$
Case **H**	$Sp_{4k(p+q)}(\mathbf{R})$	$O^*(2k)$	U_k	$U(2p,2q)$	$Sp(p,q)$

a symmetric pair in all the three cases.

Let $\mathfrak{g}_0$ be the Lie algebra of G_0 and $\mathfrak{g}$ its complexification, and so on. Denote by ω the Weil representation (the oscillator representation) of $\mathfrak{sp}_{2N}$, where $\mathfrak{sp}_{2N}$ is the complexified Lie algebra of $Sp_{2N}(\mathbf{R})$. Then we have the following equation in the Weyl algebra on $V \simeq \mathbf{C}^N$:

$$\omega(U(\mathfrak{g})^K) = \omega(U(\mathfrak{m})^H),$$

where K and H denote the complexifications of K_0 and H_0 respectively, and $U(\mathfrak{g})^K$ denotes the set of K-invariants in the universal enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. The subalgebra $S(\mathfrak{p})^K$ of the K-invariants in the symmetric algebra $S(\mathfrak{p})$ is isomorphic to a polynomial ring, and let $X_1, X_2, \dots, X_r$ be a set of generators of $S(\mathfrak{p})^K$. Let us take a K-linear mapping $\iota : S(\mathfrak{p}) \to U(\mathfrak{g})$. The image $\iota(X_d)$ is K-invariant and hence $\omega(\iota(X_d))$ can be expressed in terms of $\omega(U(\mathfrak{m})^H)$:

$$\omega(\iota(X_d)) = \omega(C_d) \qquad\qquad (C_d \in U(\mathfrak{m})^H).$$

We call this formula *a Capelli identity for a symmetric pair* and C_d *a Capelli element for a symmetric pair.*

The Capelli identity depends on the choice of the K-linear mapping ι. We take ι as follows. Let $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$ be the irreducible decomposition of a K-module. Then both $\mathfrak{p}^-$ and $\mathfrak{p}^+$ are commutative Lie algebras, since $\mathfrak{g}_0$ is of Hermitian type. We therefore have the isomorphism,

$$S(\mathfrak{p}) \simeq S(\mathfrak{p}^+) \otimes_{\mathbf{C}} S(\mathfrak{p}^-) = U(\mathfrak{p}^+) \otimes_{\mathbf{C}} U(\mathfrak{p}^-),$$

and define $\iota : S(\mathfrak{p}) \to U(\mathfrak{g})$ by the composite of this isomorphism and the multiplication $u_1 \otimes u_2 \mapsto u_1 u_2$ on $U(\mathfrak{g})$,

$$\iota(u_1 u_2) = u_1 u_2 \qquad\qquad (u_1 \in S(\mathfrak{p}^+), u_2 \in S(\mathfrak{p}^-)). \tag{1.1}$$

This ι satisfies $(*)$: $\mathrm{gr}_i(\iota(u)) = u$ for every homogeneous element $u \in S^i(\mathfrak{p})$, where $\mathrm{gr}_i : F_i U(\mathfrak{g}) \to S^i(\mathfrak{g})$ is the canonical map from the subspace $F_i U(\mathfrak{g})$ of filter degree i of the filtered algebra $U(\mathfrak{g})$ to the homogeneous subspace $S^i(\mathfrak{g})$ of degree i of the graded algebra $S(\mathfrak{g})$. We call a K-map satisfying $(*)$ *a pseudo-symmetrization map.*

We give the Capelli identities only when M_0 is compact, that is, $M_0 = U_n$ or $U_n \times U_n$ in this article. We, however, strongly believe that we can obtain the Capelli identities for the cases where M_0 is not compact by using the Fourier transform of the Weyl algebra on V, due to the suggestion of Hiroyuki Ochiai and Jiro Sekiguchi.

This work is motivated by the harmonic analysis on symmetric spaces [Hua02], [Lee04], for instance. We discuss an application to the harmonic analysis in a forthcoming paper.

2 Case R

In this section, we give the Capelli identity for the symmetric pair of Case **R** in Table 1. We first fix the notation, describe the generators of $S(\mathfrak{p})^K$, and state the main theorem for Case **R**. Before proving the theorem, we demonstrate the computation when taking the principal symbols, in order to see the outline of the proof. We prove two key lemmas and they complete the proof of the theorem. One of these lemmas is also used for Case **C** and Case **H**. At the end of this section, we prove that the Capelli elements are H-invariant.

2.1 Preliminary

Define a complex Lie algebra $\mathfrak{g}$, its subalgebras $\mathfrak{k}$ and $\mathfrak{p}^{\pm}$, and elements of these algebras.

$$\mathfrak{g} = \mathfrak{sp}_{2k} = \left\{ \begin{pmatrix} H & G \\ F & -{}^tH \end{pmatrix} \;\middle|\; \begin{array}{l} H \in \mathfrak{gl}_k, \\ G, F \in \mathrm{Sym}(k; \mathbf{C}) \end{array} \right\}, \quad \mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$
$$\mathfrak{k} = \left\{ \begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_k, \qquad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$

$$H_{ij} = E_{ij} - E_{k+j,k+i} \in \mathfrak{k}, \quad G_{ij} = E_{i,k+j} + E_{j,k+i} \in \mathfrak{p}^+, \quad F_{ij} = E_{k+i,j} + E_{k+j,i} \in \mathfrak{p}^-,$$

where E_{ij} denotes the matrix unit and $\mathrm{Sym}(k; \mathbf{C})$ denotes the set of the complex symmetric $k \times k$ matrices. Define a complex Lie algebra $\mathfrak{m}$ and its subalgebra $\mathfrak{h}$ by

$$\mathfrak{m} = \mathfrak{gl}_n, \qquad \mathfrak{h} = \mathfrak{o}_n = \{X \in \mathfrak{gl}_n \;;\; {}^tX + X = 0_n\}.$$

Set $V = \mathrm{Mat}(n, k; \mathbf{C})$ and denote the linear coordinate functions on V and the corresponding differential operators by

$$x_{si}, \partial_{si} \quad (1 \leq s \leq n, 1 \leq i \leq k),$$

respectively.

Let $\mathfrak{s} = \mathfrak{sp}_{2kn}$ be the complex symplectic Lie algebra, in which both $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{m})$ form dual pairs. We have the Weil representation ω of $\mathfrak{s}$ on the space $\mathbf{C}[V]$ of polynomial functions on V, and its explicit forms on $\mathfrak{g}$ and $\mathfrak{m}$ are as follows:

$$\begin{aligned} \omega(G_{ij}) &= \sqrt{-1} \sum_{s=1}^{n} x_{si} x_{sj}, & \omega(F_{ij}) &= \sqrt{-1} \sum_{s=1}^{n} \partial_{si} \partial_{sj}, \\ \omega(H_{ij}) &= \sum_{s=1}^{n} x_{si} \partial_{sj} + \frac{n}{2} \delta_{ij}, & \omega(E_{st}) &= \sum_{i=1}^{k} x_{si} \partial_{ti} + \frac{k}{2} \delta_{st}. \end{aligned} \tag{2.1}$$

We now recall the structure of $S(\mathfrak{p})^K$. Since $\mathfrak{g}_0$ is of Hermitian type, $K \simeq GL_k(\mathbf{C})$ acts multiplicity-freely both on the symmetric algebra $S(\mathfrak{p}^+)$ and on $S(\mathfrak{p}^-)$:

$$S(\mathfrak{p}^+) = \bigoplus_{\mu} W_\mu, \qquad S(\mathfrak{p}^-) = \bigoplus_{\mu} W_\mu^*,$$

where μ runs over the set of all the even partitions with length at most k, W_μ is the simple $\mathfrak{k}$-submodule of $S(\mathfrak{p}^+)$ parametrized by the partition μ, and W_μ^* is the simple submodule of $S(\mathfrak{p}^-)$ dual to W_μ. Thus we have the expression of $S(\mathfrak{p})^K$,

$$S(\mathfrak{p})^K = (S(\mathfrak{p}^+) \otimes_{\mathbf{C}} S(\mathfrak{p}^-))^K = \bigoplus_{\mu} (W_\mu \otimes_{\mathbf{C}} W_\mu^*)^K.$$

In fact, $S(\mathfrak{p})^K$ is isomorphic to a polynomial ring with k algebraically independent generators. For $d = 1, 2, \ldots k$, the dth generator is the basis vector of the one-dimensional vector space $(W_\mu \otimes_{\mathbf{C}} W_\mu^*)^K$ for $\mu = (2, 2, \ldots, 2, 0, \ldots, 0)$ where, 2 appears d times and 0 appears $(k - d)$ times. The explicit form of the generators are

$$X_d = \sum_{I,J \in \mathcal{I}_d^k} \det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI} \in S(\mathfrak{p})^K \qquad (d = 1, 2, \ldots, k = r), \tag{2.2}$$

where $\mathcal{I}_d^k$ is the index set defined by $\{I \subset \{1, 2, \ldots, k\} \mid \#I = d\}$, and $\mathbf{G}_{IJ}$ denotes the $d \times d$ submatrix of the $k \times k$ matrix (G_{ij}) with the rows and the columns chosen by I and J, respectively. Note that the generators above belong to the symmetric algebra $S(\mathfrak{p})$, and that G_{ij} and $F_{i'j'}$ appearing in the generators commute with each other in this context.

2.2 Capelli identity for Case R

The pseudo-symmetrization map ι defined by (1.1) embeds the generators (2.2) of $S(\mathfrak{p})^K$ into $U(\mathfrak{g})$ without symmetrization. Hence the image of the generator X_d under ι looks the same as X_d itself, except that the images are in $U(\mathfrak{g})$. In the following theorem, we use the column-determinant for the determinant of a matrix with non-commutative entries, defined by

$$\det(Z_{ij}) = \sum_{\sigma \in \mathfrak{S}_d} Z_{\sigma(1)1} Z_{\sigma(2)2} \cdots Z_{\sigma(d)d}.$$

Theorem 2.1. *For $1 \le d \le \min(k, n)$, we have the Capelli identities for the symmetric pair of Case* **R** *in Table 1:*

$$\omega\Big(\sum_{I,J\in\mathcal{I}_d^k} \det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}\Big)$$
$$= \omega\Big((-1)^d \sum_{S,T\in\mathcal{I}_d^n} \det(E_{S(i)T(j)} + (d-j-1-k/2)\delta_{S(i),T(j)})_{i,j}$$
$$\times \det(E_{S(i)T(j)} + (d-j-k/2)\delta_{S(i),T(j)})_{i,j}\Big),$$

where $S(i)$ denotes an element of the index set S with $S(1) < S(2) < \cdots < S(d)$. The expression on the right-hand side is the image under ω of a sum of products of two $d \times d$ minors with entries in $U(\mathfrak{m})$.

Note that $\sum_{I,J} \det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}$ on the left-hand side is the image under ι of the generator (2.2), and that it is an element of $U(\mathfrak{g})$ in particular. There are k generators of $S(\mathfrak{p})^K$ as (2.2), however the equation above is trivial when $n < d \le k$ since the right-hand side becomes an empty sum. □

As explained in Introduction, the right-hand side of the Capelli identity is H-invariant in the Weyl algebra, however it is not automatic that its inverse image is H-invariant in $U(\mathfrak{m})$. In fact, the inverse image is H-invariant and we prove this invariance at the end of this section.

Before proving Theorem 2.1, we demonstrate the computation when taking the principal symbols, in order to see the outline of the proof. This computation forms a part of the proof of the theorem. We first recall a basic lemma.

Lemma 2.2 (Cauchy-Binet). *Let R be a commutative ring and $d \le N$. For $A \in \mathrm{Mat}(d, N; R)$ and $B \in \mathrm{Mat}(N, d; R)$, we have*

$$\det AB = \sum_{S\in\mathcal{I}_d^N} \det A_{\bullet,S} \det B_{S,\bullet},$$

where $A_{\bullet,S}$ is the $d \times d$ submatrix of A in which all the rows are chosen and the columns are chosen by S. □

Define $n \times k$ matrices X and ∂ by

$$X = (x_{si})_{1\le s\le n,\ 1\le i\le k}, \qquad \partial = (\partial_{si})_{1\le s\le n,\ 1\le i\le k}.$$

In the following computation we take the principal symbols, and we write the principal symbol of ∂_{si} by the same letter ∂_{si}. For $I, J \in \mathcal{I}_d^k$, the lemma above yields

$$\begin{aligned}\omega(\det(\mathbf{G}_{IJ})) &= \det\left(\sqrt{-1}\sum_{s=1}^{n} x_{s,I(i)}x_{s,J(j)}\right)_{1\le i,j\le d}\\ &= (\sqrt{-1})^d \det({}^t(X_{\bullet,I})X_{\bullet,J})\\ &= (\sqrt{-1})^d \sum_{S\in\mathcal{I}_d^n} \det {}^t(X_{SI}) \det X_{SJ}.\end{aligned}$$

Similarly we have

$$\omega(\det(\mathbf{F}_{JI})) = (\sqrt{-1})^d \sum_{T\in\mathcal{I}_d^n} \det {}^t(\partial_{TJ}) \det \partial_{TI},$$

and the equation of matrices

$$\big(\omega(E_{S(i)T(j)})\big)_{1\le i,j\le d} = (X\,{}^t\partial)_{ST} \qquad (S,T\in\mathcal{I}_d^n),$$

where $E_{S(i)T(j)}$ is an element in $\mathfrak{m}$. Note here that the contribution of the character appearing in (2.1) vanishes, since we are taking the principal symbols. Note also that elements in the expressions above commute with each other for the same reason, and we have

$$\begin{aligned}\sum_{I,J\in\mathcal{I}_d^k} \omega(\det\mathbf{G}_{IJ}\cdot\det\mathbf{F}_{JI}) &= (-1)^d \sum_{I,J}\sum_{S,T\in\mathcal{I}_d^n} \det {}^t(X_{SI}) \det X_{SJ} \det {}^t(\partial_{TJ}) \det \partial_{TI}\\ &=^{*} (-1)^d \sum_{I,S,T} \det {}^t(X_{SI}) \det(X\,{}^t\partial)_{ST} \det \partial_{TI}\\ &=^{**} (-1)^d \sum_{I,S,T} \det(X\,{}^t\partial)_{ST} \det {}^t(X_{SI}) \det \partial_{TI}\\ &=^{*} (-1)^d \sum_{S,T} \det(X\,{}^t\partial)_{ST} \det(X\,{}^t\partial)_{ST}\\ &= (-1)^d \sum_{S,T} \omega(\det\mathbf{E}_{ST}\det\mathbf{E}_{ST}).\end{aligned} \tag{2.3}$$

This is nothing but our desired formula of Theorem 2.1 except that there are no diagonal shifts in the last expression above. The equalities with $*$ and $**$ above do not hold when we do *not* take the principal symbols, and we prove the non-commutative analogues of these two equalities with diagonal shifts in the following subsections.

Remark 2.3. First, the non-commutative analogue of the equality with $*$ is, in fact, the formula which is used for proving the classical Capelli identity. So the formula is known, and there is essentially the same formula in [Ume, Theorem 2], for instance.

Second, the non-commutative analogue of the equality with $**$ seems a natural formula when considering the prehomogeneous vector space $(\mathfrak{gl}_n, V)$.

2.3 First lemma for the theorem

We prove the non-commutative analogue of the equality with $*$ in (2.3). Remark that the goal of this subsection, Lemma 2.6, is not new as mentioned in Remark 2.3, however we give a complete proof using the exterior algebra. This method is very effective to simplify the computation involving determinants or permanents, and has been used mainly for constructing central elements of universal enveloping algebras of simple Lie algebras [IU01], [Wac03], [Ito04b], and for obtaining Capelli identities of various types [Ume], [Ito03], [Ito04a], [Wac04].

In this subsection we fix $S, T \in \mathcal{I}_d^n$.

Definition 2.4. Let $e_1, e_2, \ldots, e_d \in \mathbf{C}^d$ be the standard basis, and form the exterior algebra $\bigwedge \mathbf{C}^d$. Define the elements η_l, ζ_j and $\zeta_j(u)$ in the tensor product algebra $\bigwedge \mathbf{C}^d \otimes_{\mathbf{C}} \mathrm{End}(\mathbf{C}[V])$ by

$$\eta_l = \sum_{i=1}^{d} e_i x_{S(i),l} \qquad (1 \le l \le k),$$

$$\zeta_j = \sum_{i=1}^{d} e_i \omega(E_{S(i)T(j)} - (k/2)\delta_{S(i),T(j)}) \qquad (1 \le j \le d,\ u \in \mathbf{C}),$$

$$\zeta_j(u) = \sum_{i=1}^{d} e_i \omega(E_{S(i)T(j)} + (u - k/2)\delta_{S(i),T(j)}) \qquad (1 \le j \le d,\ u \in \mathbf{C}).$$

Note that products of these elements produces determinants. For example,

$$\eta_{I(1)}\eta_{I(2)} \cdots \eta_{I(d)} = e_1 e_2 \cdots e_d \det X_{SI}.$$

Lemma 2.5. *We have the following relations:*
(1) $\zeta_j = \sum_{l=1}^{k} \eta_l \partial_{T(j),l} \qquad (1 \le j \le d)$,
(2) $\zeta_j(u)\eta_m = -\eta_m \zeta_j(u-1) \qquad (1 \le j \le d,\ 1 \le m \le k)$.

Proof. (1) $\zeta_j = \sum_{i=1}^{d} e_i \sum_{l=1}^{k} x_{S(i),l}\partial_{T(j),l} = \sum_l \eta_l \partial_{T(j),l}$. For (2), we compute as follows:

$$\begin{aligned}\zeta_j \eta_m &= \sum_{l=1}^{k} \eta_l \partial_{T(j),l} \sum_{i=1}^{d} e_i x_{S(i),m} \\ &= \sum_{l,i} \eta_l e_i (x_{S(i),m}\partial_{T(j),l} + \delta_{T(j),S(i)}\delta_{lm}) \qquad = -\eta_m \zeta_j + \sum_i \eta_m e_i \delta_{T(j),S(i)}.\end{aligned}$$

We add $u \sum_{i=1}^{d} e_i \delta_{S(i),T(j)} \eta_m$ to both sides of the expression above, and we obtain $\zeta_j(u)\eta_m = -\eta_m \zeta_j(u-1)$. □

Lemma 2.6. *We have the following equation:*

$$\sum_{J\in\mathcal{I}_d^k} \det X_{SJ} \det \partial_{TJ} = \det\big(\omega(E_{S(i)T(j)} + (d-j-k/2)\delta_{S(i),T(j)})\big)_{1\le i,j\le d}.$$

Proof. First we have $\zeta_j(u) = \sum_{i=1}^d e_i\omega(E_{S(i)T(j)} + (u-k/2)\delta_{S(i),T(j)})$ from the definition, and we therefore obtain

$$\zeta_1(u_1)\zeta_2(u_2)\cdots\zeta_d(u_d) = e_1e_2\cdots e_d \det(\omega(E_{S(i)T(j)} + (u_j - k/2)\delta_{S(i),T(j)}))_{1\le i,j\le d},$$

for $u_i \in \mathbf{C}$.

Second, using Lemma 2.5 (1) and (2) repeatedly, we have

$$\begin{aligned}\zeta_1(d-1)\zeta_2(d-2)\cdots\zeta_d(0) &= \zeta_1(d-1)\cdots\zeta_{d-1}(1)\sum_{l=1}^k \eta_l\partial_{T(d),l}\\ &= (-1)^{d-1}\sum_{l=1}^k \eta_l\cdot\zeta_1(d-2)\cdots\zeta_{d-1}(0)\cdot\partial_{T(d),l}\\ &\ \ \vdots\\ &= ((-1)^{d-1})^d \sum_{l_1,\dots,l_d=1}^k \eta_{l_1}\cdots\eta_{l_d}\cdot\partial_{T(1),l_1}\cdots\partial_{T(d),l_d}.\end{aligned}$$

Since η_{l_j}'s are anti-commutative (i.e. $\eta_{l_j}\eta_{l_{j'}} + \eta_{l_{j'}}\eta_{l_j} = 0$), l_j's are distinct. Hence the expression above equals

$$\begin{aligned}&\sum_{J\in\mathcal{I}_d^k}\sum_{\sigma\in\mathfrak{S}_d} \eta_{J(\sigma(1))}\cdots\eta_{J(\sigma(d))}\cdot\partial_{T(1),J(\sigma(1))}\cdots\partial_{T(d),J(\sigma(d))}\\ &\quad= \sum_{J,\sigma}\eta_{J(1)}\cdots\eta_{J(1)}\operatorname{sgn}(\sigma)\cdot\partial_{T(1),J(\sigma(1))}\cdots\partial_{T(d),J(\sigma(d))}\\ &\quad= \sum_J e_1\cdots e_d\det X_{SJ}\cdot\det\partial_{TJ}.\end{aligned}$$

Comparing these two formulas we have the lemma. □

2.4 Second lemma for the theorem

We prove the non-commutative analogue of the equality with ** in (2.3). Remark that the goal of this subsection, Lemma 2.9, is not new as mentioned in Remark 2.3, however we give a complete proof using the exterior algebra again.

In this subsection we fix $S, T \in \mathcal{I}_d^n$ and $I \in \mathcal{I}_d^k$.

Definition 2.7. Define the elements η'_i, μ_j and $\mu_j(u)$ in the tensor product algebra $\bigwedge \mathbf{C}^d \otimes_{\mathbf{C}} \mathrm{End}(\mathbf{C}[V])$ by

$$\eta'_i = \sum_{h=1}^{d} e_h x_{S(i),I(h)} \qquad (1 \le i \le d),$$

$$\mu_j = \sum_{i=1}^{d} \eta'_i \omega(E_{S(i)T(j)} - (k/2)\delta_{S(i),T(j)}) \qquad (1 \le j \le d,\ u \in \mathbf{C}).$$

$$\mu_j(u) = \sum_{i=1}^{d} \eta'_i \omega(E_{S(i)T(j)} + (u - k/2)\delta_{S(i),T(j)}) \qquad (1 \le j \le d,\ u \in \mathbf{C}).$$

Lemma 2.8. *We have the following relations:*
(1) $\mu_j \eta'_g = -\eta'_g \mu_j \qquad (1 \le j, g \le d)$,
(2) $\mu_j(u) = \sum_{i=1}^{d} \omega(E_{S(i)T(j)} + (u - 1 - k/2)\delta_{S(i),T(j)})\eta'_i \qquad (1 \le j \le d,\ u \in \mathbf{C})$.

Proof. We have (1) by a direct computation:

$$\begin{aligned}
\mu_j \eta'_g &= \sum_{i=1}^{d} \eta'_i \omega(E_{S(i)T(j)} - (k/2)\delta_{S(i),T(j)})\eta'_g \\
&= \sum_{i} \eta'_i \sum_{l=1}^{k} x_{S(i),l}\partial_{T(j),l} \sum_{h=1}^{d} e_h x_{S(g)I(h)} \\
&= \sum_{i,l,h} \eta'_i e_h x_{S(i),l}(x_{S(g)I(h)}\partial_{T(j),l} + \delta_{S(g)T(j)}\delta_{l,I(h)}) \\
&= -\eta'_g \mu_j + \sum_{i} \delta_{S(g),T(j)} \eta'_i \eta'_i \qquad = -\eta'_g \mu_j.
\end{aligned}$$

We have (2) also by a direct computation:

$$\begin{aligned}
\mu_j(u) &= \sum_{i=1}^{d} \eta'_i \omega(E_{S(i)T(j)} + (u - k/2)\delta_{S(i),T(j)}) \\
&= \sum_{i} \sum_{h=1}^{d} e_h x_{S(i)I(h)} \left(\sum_{l=1}^{k} x_{S(i),l}\partial_{T(j),l} + u\delta_{S(i),T(j)} \right) \\
&= \sum_{i,h,l} e_h x_{S(i),l}(\partial_{T(j),l} x_{S(i)I(h)} - \delta_{S(i),T(j)}\delta_{l,I(h)}) + \sum_{i} \delta_{S(i),T(j)}\eta'_i \\
&= \sum_{i} \omega(E_{S(i)T(j)} - (k/2)\delta_{S(i),T(j)})\eta'_i - \sum_{i} \delta_{S(i),T(j)}\eta'_i + u \sum_{i} \delta_{S(i),T(j)}\eta'_i \\
&= \sum_{i} \omega(E_{S(i)T(j)} + (u - 1 - k/2)\delta_{S(i),T(j)})\eta'_i.
\end{aligned}$$

$\square$

Lemma 2.9. *We have the following formula for* $u_1, u_2, \dots, u_d \in \mathbf{C}$*:*

$$\det X_{SI} \det \left(\omega(E_{S(i)T(j)} + u_j \delta_{S(i),T(j)})\right)_{1\le i,j\le d} = \det \left(\omega(E_{S(i)T(j)} + (u_j - 1)\delta_{S(i),T(j)})\right)_{1\le i,j\le d} \det X_{SI}.$$

Proof. We compute $\mu_1(u_1)\cdots\mu_d(u_d)$ in two different ways. First the factor $\mu_j(u)$ is equal to $\sum_{i=1}^d \eta_i' \omega(E_{S(i)T(j)} + (u - k/2)\delta_{S(i),T(j)})$ by the definition, and hence we have from Lemma 2.8 (1),

$$\begin{aligned}
&\mu_1(u_1)\cdots\mu_d(u_d) \\
&= \mu_1(u_1)\cdots\mu_{d-1}(u_{d-1}) \sum_{i=1}^d \eta_i' \omega(E_{S(i)T(d)} + (u_d - k/2)\delta_{S(i)T(d)}) \\
&= (-1)^{d-1} \sum_i \eta_i' \cdot \mu_1(u_1)\cdots\mu_{d-1}(u_{d-1}) \cdot \omega(E_{S(i)T(d)} + (u_d - k/2)\delta_{S(i)T(d)}) \\
&\vdots \\
&= ((-1)^{d-1})^d \sum_{i_1,\dots,i_d=1}^d \eta_{i_1}' \cdots \eta_{i_d}' \cdot \omega(E_{S(i_1)T(1)} + (u_1 - k/2)\delta_{S(i_1)T(1)}) \cdots \\
&\qquad \cdots \omega(E_{S(i_d)T(d)} + (u_d - k/2)\delta_{S(i_d)T(d)}).
\end{aligned}$$

Since η_{i_j}''s are anti-commutative, i_j's are distinct in the above expression. Similarly to the proof of Lemma 2.6, the expression above is thus equal to

$$e_1 \cdots e_d \det X_{SI} \cdot \det(\omega(E_{S(i)T(j)} + (u_j - k/2)\delta_{S(i),T(j)})).$$

Second we compute $\mu_1(u_1)\cdots\mu_d(u_d)$ in another way. It follows from Lemma 2.8 (2) that $\mu_j(u) = \sum_{i=1}^d \omega(E_{S(i)T(j)} + (u - 1 - k/2)\delta_{S(i),T(j)})\eta_i'$. So this time we move η''s to the right in the product $\mu_1(u_1)\cdots\mu_d(u_d)$, and we have

$$\mu_1(u_1)\cdots\mu_d(u_d) = e_1\cdots e_d \det(\omega(E_{S(i)T(j)} + (u_j - 1 - k/2)\delta_{S(i),T(j)})) \cdot \det X_{SI}.$$

Comparing these two computations we have the lemma. $\square$

2.5 Proof of Theorem 2.1

To begin with the first equality of (2.3), using Lemma 2.6 and Lemma 2.9, we can immediately prove Theorem 2.1. We first have

$$\sum_{I,J\in\mathcal{I}_d^k} \omega(\det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}) = (-1)^d \sum_{I,J} \sum_{S,T\in\mathcal{I}_d^{2n}} \det {}^t(X_{SI}) \det X_{SJ} \det {}^t(\partial_{TJ}) \det \partial_{TI},$$

from the first equality of (2.3). It follows from Lemma 2.6 that this is equal to

$$(-1)^d \sum_{I,S,T} \det {}^t(X_{SI}) \det \big(\omega(E_{S(i)T(j)} + (d-j-k/2)\delta_{S(i),T(j)})\big)_{1\le i,j\le d} \det \partial_{TI}.$$

By Lemma 2.9, it turns out that the expression above equals

$$(-1)^d \sum_{I,S,T} \det \big(\omega(E_{S(i)T(j)} + (d-j-1-k/2)\delta_{S(i),T(j)})\big)_{1\le i,j\le d} \cdot \det {}^t(X_{SI}) \det \partial_{TI}.$$

By using Lemma 2.6 again, this is equal to

$$(-1)^d \sum_{S,T} \det \big(\omega(E_{S(i)T(j)} + (d-j-1-k/2)\delta_{S(i),T(j)})\big)_{1\le i,j\le d}$$
$$\times \det \big(\omega(E_{S(i)T(j)} + (d-j-k/2)\delta_{S(i),T(j)})\big)_{1\le i,j\le d}.$$

We thus have proved Theorem 2.1.

2.6 Invariance of the Capelli elements

We call the following element $C_d^{\mathbf{R}}$ *the Capelli element for the symmetric pair*, which is the element appearing on the right-hand side of the formula of Theorem 2.1:

$$C_d^{\mathbf{R}} = (-1)^d \sum_{S,T\in\mathcal{I}_d^n} \det(E_{S(i)T(j)} + (d-j-1-k/2)\delta_{S(i),T(j)})_{i,j}$$
$$\times \det(E_{S(i)T(j)} + (d-j-k/2)\delta_{S(i),T(j)})_{i,j},$$

for $d = 1, 2, \ldots, n$. Note that the image of the Capelli element $C_d^{\mathbf{R}}$ under the Weil representation ω is zero when $k < d \le n$, since the left-hand side of the Capelli identity in Theorem 2.1 becomes an empty sum, while $C_d^{\mathbf{R}} \ne 0$ for $1 \le d \le n$. The Capelli element $C_d^{\mathbf{R}}$ is not a central element of $U(\mathfrak{m})$, but an H-invariant element.

Proposition 2.10. *The Capelli element is H-invariant, that is, $C_d^{\mathbf{R}} \in U(\mathfrak{m})^H$, where $H = O_n(\mathbf{C})$.*

Proof. We just give an outline of the proof. Consider the tensor product algebra $W = \bigwedge \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge \mathbf{C}^n \otimes_{\mathbf{C}} U(\mathfrak{m})$. Denote the standard basis of $\mathbf{C}^n$ in the first and the second factor by e_t and e'_t respectively. Define $\eta_t(u)$ and $\eta'_t(u)$ in W by

$$\eta_t(u) = \sum_{s=1}^n e_s(E_{st} + u\delta_{st}), \qquad \eta'_t(u) = \sum_{s=1}^n e'_s(E_{st} + u\delta_{st}),$$

Note that we have for $u, v \in \mathbf{C}$

$$\eta_T(u) = \sum_{S \in \mathcal{I}_d^n} e_S \det \mathbf{E}_{ST}(u), \qquad \eta'_T(v) = \sum_{S \in \mathcal{I}_d^n} e'_S \det \mathbf{E}_{ST}(v) \qquad (T \in \mathcal{I}_d^n),$$

where $\eta_T(u) = \eta_{T(1)}(u-1)\eta_{T(2)}(u-2)\cdots\eta_{T(d)}(u-d)$, $e_S = e_{S(1)}e_{S(2)}\cdots e_{S(d)}$ and $\mathbf{E}_{ST}(u)$ denotes the $d \times d$ matrix whose (i,j)-entry is $E_{S(i)T(j)} + (u-j)\delta_{S(i),T(j)}$.

Now we give an M-module structure, that is, a $GL_n(\mathbf{C})$-module structure to W. First, $U(\mathfrak{m}) = U(\mathfrak{gl}_n)$ is a $GL_n(\mathbf{C})$-module through the adjoint action. Second, for both $\mathbf{C}^n$, we give the module structure dual to the natural representation of $GL_n(\mathbf{C})$. The tensor product W thus has a $GL_n(\mathbf{C})$-module structure. Let W_d be the submodule of W spanned by $e_T e'_{T'}$ with $T, T' \in \mathcal{I}_d^n$. Then it is known that the mapping

$$\begin{array}{cccc} \Delta: & W_d & \to & W \\ & e_T e'_{T'} & \mapsto & \eta_T(u)\eta'_{T'}(v) \end{array} \qquad (T, T' \in \mathcal{I}_d^n)$$

is a $GL_n(\mathbf{C})$-homomorphism for $u, v \in \mathbf{C}$, and it is $O_n(\mathbf{C})$-homomorphism in particular. The $GL_n(\mathbf{C})$-module $\mathbf{C}^n$ is also a self-dual $O_n(\mathbf{C})$-module through the restricted action, and the element $\sum_T e_T e'_T \in W_d$ is $O_n(\mathbf{C})$-invariant. We can define the contraction mapping ε on W_d by $\varepsilon(e_T e'_{T'}) = \delta_{T,T'}$, which is naturally extended to $W_d \otimes_{\mathbf{C}} U(\mathfrak{m})$, and it is an $O_n(\mathbf{C})$-homomorphism.

We can prove the assertion by using the $O_n(\mathbf{C})$-homomorphisms Δ and ε.

$$\begin{aligned} \varepsilon\left(\Delta\left(\sum_T e_T e'_T\right)\right) &= \varepsilon\left(\sum_T \eta_T(u)\eta'_T(v)\right) \\ &= \varepsilon\left(\sum_{T,S,S'} e_S \det \mathbf{E}_{ST}(u) e'_{S'} \det \mathbf{E}_{S'T}(v)\right) \\ &= \sum_{S,T} \det \mathbf{E}_{ST}(u) \det \mathbf{E}_{ST}(v). \end{aligned}$$

The last expression is also $O_n(\mathbf{C})$-invariant, since $\sum e_T e'_T$ is $O_n(\mathbf{C})$-invariant. We thus have proved the assertion. $\square$

3 Case C

In this section, we give the Capelli identity for the symmetric pair of Case $\mathbf{C}$ in Table 1. We first fix the notation, describe the generators of $S(\mathfrak{p})^K$, and then prove the main theorem for Case $\mathbf{C}$.

3.1 Preliminary

Define complex Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}^\pm$ and elements of these algebras by

$$\mathfrak{g} = \mathfrak{gl}_{p+q} = \left\{ \begin{pmatrix} H^{(x)} & G \\ F & H^{(y)} \end{pmatrix} \;\middle|\; \begin{matrix} H^{(x)} \in \mathfrak{gl}_p, & G \in \mathrm{Mat}(p,q;\mathbf{C}), \\ H^{(y)} \in \mathfrak{gl}_q, & F \in \mathrm{Mat}(q,p;\mathbf{C}) \end{matrix} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} H^{(x)} & 0 \\ 0 & H^{(y)} \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_p \oplus \mathfrak{gl}_q,$$

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\}, \qquad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$

$$H^{(x)}_{ij} = E_{ij} \in \mathfrak{k} \quad (1 \le i,j \le p), \quad G_{ij} = E_{i,p+j} \in \mathfrak{p}^+ \quad (1 \le i \le p, 1 \le j \le q),$$
$$H^{(y)}_{ij} = E_{p+i,p+j} \in \mathfrak{k} \quad (1 \le i,j \le q), \quad F_{ij} = E_{p+i,j} \in \mathfrak{p}^- \quad (1 \le i \le q, 1 \le j \le p).$$

Define a complex Lie algebra $\mathfrak{m}$, its subalgebra $\mathfrak{h}$, and elements of $\mathfrak{m}$ by

$$\mathfrak{m} = \mathfrak{gl}_n \oplus \mathfrak{gl}_n, \qquad \mathfrak{h} = \{(X, -{}^tX) \in \mathfrak{m}\} \simeq \mathfrak{gl}_n,$$

$$E^{(x)}_{st} = (E_{st}, 0) \in \mathfrak{m}, \qquad E^{(y)}_{st} = (0, E_{st}) \in \mathfrak{m} \qquad (1 \le s,t \le n).$$

Set $V = \mathrm{Mat}(n,p;\mathbf{C}) \oplus \mathrm{Mat}(n,q;\mathbf{C})$ and denote the linear coordinate functions on each component of V by

$$x_{si}, y_{sj} \qquad (1 \le s \le n,\ 1 \le i \le p,\ 1 \le j \le q),$$

respectively.

Set $\mathfrak{s} = \mathfrak{sp}_{2(p+q)n}$ in which both $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{m})$ form dual pairs. We have the Weil representation ω of $\mathfrak{s}$ on $\mathbf{C}[V]$, and its explicit forms on $\mathfrak{g}$ and $\mathfrak{m}$ are as follows:

$$\omega(H^{(x)}_{ij}) = \sum_{s=1}^n x_{si}\frac{\partial}{\partial x_{sj}} + \frac{n}{2}\delta_{ij}, \qquad \omega(H^{(y)}_{ij}) = -\sum_{s=1}^n y_{sj}\frac{\partial}{\partial y_{si}} - \frac{n}{2}\delta_{ij},$$
$$\omega(G_{ij}) = \sqrt{-1}\sum_{s=1}^n x_{si}y_{sj}, \qquad \omega(F_{ji}) = \sqrt{-1}\sum_{s=1}^n \frac{\partial}{\partial x_{si}}\frac{\partial}{\partial y_{sj}},$$
$$\omega(E^{(x)}_{st}) = \sum_{i=1}^p x_{si}\frac{\partial}{\partial x_{ti}} + \frac{p}{2}\delta_{st}, \qquad \omega(E^{(y)}_{st}) = \sum_{j=1}^q y_{sj}\frac{\partial}{\partial y_{tj}} + \frac{q}{2}\delta_{st}.$$

We now recall the structure of $S(\mathfrak{p})^K$. Similarly to Case $\mathbf{R}$, we have the decomposition of $S(\mathfrak{p})^K$,

$$S(\mathfrak{p})^K = \bigoplus_\mu (W_\mu \otimes_{\mathbf{C}} W^*_\mu)^K,$$

where W_μ and W_μ^* are the simple submodules of $S(\mathfrak{p}^+)$ and $S(\mathfrak{p}^-)$ respectively, they are dual to each other, and μ runs over the set of all the partitions with length at most $\min(p,q)$. In fact, $S(\mathfrak{p})^K$ is isomorphic to a polynomial ring with $\min(p,q)$ algebraically independent generators, and their explicit forms are

$$X_d = \sum_{I\in\mathcal{I}_d^p, J\in\mathcal{I}_d^q} \det \mathbf{G}_{IJ}\cdot\det\mathbf{F}_{JI} \qquad (d=1,2\dots,r;\quad r=\min(p,q)). \tag{3.1}$$

Note that the generators above belong to the symmetric algebra $S(\mathfrak{p})$, and that G_{ij} and $F_{i'j'}$ appearing in the generators commute with each other in this context.

3.2 Capelli identity for Case C

Theorem 3.1. *For $1\le d\le \min(p,q,n)$, we have the Capelli identities for the symmetric pair of Case* **C** *in Table 1:*

$$\begin{aligned}\omega&\Bigl(\sum_{I\in\mathcal{I}_d^p, J\in\mathcal{I}_d^q} \det \mathbf{G}_{IJ}\cdot\det\mathbf{F}_{JI}\Bigr)\\ &=\omega\Bigl((-1)^d\sum_{S,T\in\mathcal{I}_d^n}\det(E^{(x)}_{S(i)T(j)}+(d-j-p/2)\delta_{S(i),T(j)})_{i,j}\\ &\qquad\qquad\times\det(E^{(y)}_{S(i)T(j)}+(d-j-q/2)\delta_{S(i),T(j)})_{i,j}\Bigr).\end{aligned}$$

The expression on the right-hand side is the image under ω of a sum of products of two $d\times d$ minors with entries in $U(\mathfrak{m})$. There are $\min(p,q)$ generators of $S(\mathfrak{p})^K$ as (3.1), however the equation above is trivial when $n<d\le\min(p,q)$ since the right-hand side becomes an empty sum.

Proof. As in §§2.2 of Case **R**, we define the matrices

$$X=(x_{si})_{1\le s\le n,1\le i\le p}, \qquad Y=(y_{sj})_{1\le s\le n,1\le j\le q},$$
$$\partial^X=\Bigl(\frac{\partial}{\partial x_{si}}\Bigr)_{1\le s\le n,1\le i\le p}, \qquad \partial^Y=\Bigl(\frac{\partial}{\partial y_{sj}}\Bigr)_{1\le s\le n,1\le j\le q},$$

and we thus have

$$\omega(\det(\mathbf{G}_{IJ}))=(\sqrt{-1})^d\sum_{S\in\mathcal{I}_d^n}\det {}^t(X_{SI})\det Y_{SJ},$$
$$\omega(\det(\mathbf{F}_{JI}))=(\sqrt{-1})^d\sum_{T\in\mathcal{I}_d^n}\det {}^t(\partial^Y_{TJ})\det \partial^X_{TI},$$

for $I \in \mathcal{I}_d^p$, $J \in \mathcal{I}_d^q$. Using these formulas, we can prove the theorem as follows:

$$
\begin{aligned}
&\sum_{I \in \mathcal{I}_d^p, J \in \mathcal{I}_d^q} \omega(\det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}) \\
&\quad = (-1)^d \sum_{I,J} \sum_{S,T \in \mathcal{I}_d^n} \det {}^t(X_{SI}) \det Y_{SJ} \det {}^t(\partial^Y_{TJ}) \det \partial^X_{TI} \\
&\quad = (-1)^d \sum_{I,J,S,T} \det X_{SI} \det \partial^X_{TI} \cdot \det Y_{SJ} \det \partial^Y_{TJ} \\
&\quad = (-1)^d \sum_{S,T \in \mathcal{I}_d^n} \omega\Big(\det(E^{(x)}_{S(i)T(j)} + (d-j-p/2)\delta_{S(i),T(j)})_{i,j} \\
&\qquad\qquad \times \det(E^{(y)}_{S(i)T(j)} + (d-j-q/2)\delta_{S(i),T(j)})_{i,j} \Big).
\end{aligned}
$$

The last equality above follows from Lemma 2.6 of Case **R** by replacing k with p or q. □

3.3 Invariance of the Capelli elements

The Capelli element is

$$
\begin{aligned}
C_d^{\mathbf{C}} = (-1)^d \sum_{S,T \in \mathcal{I}_d^n} & \det(E^{(x)}_{S(i)T(j)} + (d-j-p/2)\delta_{S(i),T(j)})_{i,j} \\
& \times \det(E^{(y)}_{S(i)T(j)} + (d-j-q/2)\delta_{S(i),T(j)})_{i,j},
\end{aligned}
$$

for $d = 1, 2, \ldots, n$, which appears on the right-hand side of the formula of Theorem 3.1. Note that $\omega(C_d^{\mathbf{C}})$ is zero when $\min(p,q) < d \leq n$, while $C_d^{\mathbf{C}} \neq 0$ for $1 \leq d \leq n$. The Capelli element $C_d^{\mathbf{C}}$ is not a central element of $U(\mathfrak{m})$, but an H-invariant element. The following proposition is proved similarly to Proposition 2.10 of Case **R**.

Proposition 3.2. *The Capelli element is H-invariant, that is, $C_d^{\mathbf{C}} \in U(\mathfrak{m})^H$, where $H \simeq GL_n(\mathbf{C})$.* □

4 Case H

In this section, we give the Capelli identity for the symmetric pair of Case **H** in Table 1. We first fix the notation, describe the generator of $S(\mathfrak{p})^K$, and then prove the main theorem for Case **H**. The proof needs a lemma due to Ishikawa-Wakayama [IW00].

4.1 Preliminary

Define complex Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}^{\pm}$ and elements of these algebras by

$$\mathfrak{g} = \mathfrak{o}_{2k} = \left\{ \begin{pmatrix} H & G \\ F & -{}^tH \end{pmatrix} \;\middle|\; H \in \mathfrak{gl}_k, G, F \in \mathrm{Alt}(k; \mathbf{C}) \right\}, \quad \mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_k, \qquad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$

$$H_{ij} = E_{ij} - E_{k+j,k+i} \in \mathfrak{k},$$
$$G_{ij} = E_{i,k+j} - E_{j,k+i} \in \mathfrak{p}^+, \qquad F_{ij} = E_{k+i,j} - E_{k+j,i} \in \mathfrak{p}^- \qquad (1 \le i, j \le k),$$

where $\mathrm{Alt}(k; \mathbf{C})$ denotes the set of the alternating $k \times k$ matrices. Define a complex Lie algebra $\mathfrak{m}$ and its subalgebra $\mathfrak{h}$ by

$$\mathfrak{m} = \mathfrak{gl}_{2n}, \qquad \mathfrak{h} = \left\{ \begin{pmatrix} H & G \\ F & -{}^tH \end{pmatrix} \;\middle|\; \begin{array}{l} H \in \mathfrak{gl}_n, \\ G, F \in \mathrm{Sym}(n; \mathbf{C}) \end{array} \right\} \simeq \mathfrak{sp}_{2n}.$$

Set $V = \mathrm{Mat}(2n, k; \mathbf{C})$ and denote the linear coordinate functions on V and the corresponding differential operators by

$$x_{si}, \partial_{si} \qquad\qquad (1 \le s \le 2n,\ 1 \le i \le k),$$

respectively.

Let $\mathfrak{s} = \mathfrak{sp}_{4kn}$ in which both $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{m})$ form dual pairs. We have the Weil representation ω of $\mathfrak{s}$ on $\mathbf{C}[V]$, and its explicit forms on $\mathfrak{g}$ and $\mathfrak{m}$ are as follows:

$$\omega(H_{ij}) = \sum_{s=1}^{2n} x_{si}\partial_{sj} + n\delta_{ij}, \qquad \omega(G_{ij}) = \sqrt{-1}\sum_{s=1}^{n}(x_{si}x_{\bar{s}j} - x_{\bar{s}i}x_{sj}),$$
$$\omega(F_{ji}) = \sqrt{-1}\sum_{s=1}^{n}(\partial_{si}\partial_{\bar{s}j} - \partial_{\bar{s}i}\partial_{sj}), \qquad \omega(E_{st}) = \sum_{i=1}^{k} x_{si}\partial_{ti} + \frac{k}{2}\delta_{st},$$

where $\bar{s} = s + n$.

We now recall the structure of $S(\mathfrak{p})^K$. Similarly to Case $\mathbf{R}$, we have the decomposition of $S(\mathfrak{p})^K$,

$$S(\mathfrak{p})^K = \bigoplus_{\mu} (W_\mu \otimes_{\mathbf{C}} W_\mu^*)^K,$$

where W_μ and W_μ^* are the simple submodules of $S(\mathfrak{p}^+)$ and $S(\mathfrak{p}^-)$ respectively, they are dual to each other, and μ runs over the set of all the partitions of the form $(\mu_1, \mu_1, \mu_2, \mu_2, \ldots)$ with length at most k. In fact, $S(\mathfrak{p})^K$ is isomorphic to a polynomial ring with $\lfloor k/2 \rfloor$ algebraically independent generators, and their explicit forms are

$$X_d = \sum_{I \in \mathcal{I}^k_{2d}} \operatorname{Pf} \mathbf{G}_{II} \cdot \operatorname{Pf} \mathbf{F}_{II} \qquad (d = 1, 2, \ldots, r; \quad r = \lfloor k/2 \rfloor). \tag{4.1}$$

where Pf denotes the Pfaffian of a alternating matrix. Note that the generators above belong to the symmetric algebra $S(\mathfrak{p})$, and that G_{ij} and $F_{i'j'}$ appearing in the generators commute with each other in this context.

4.2 Capelli identity for Case H

Theorem 4.1. *For $1 \leq d \leq \min(\lfloor k/2 \rfloor, n)$, we have the Capelli identities for the symmetric pair of Case* **H** *in Table 1:*

$$\omega\Big(\sum_{I \in \mathcal{I}^k_{2d}} \operatorname{Pf} \mathbf{G}_{II} \cdot \operatorname{Pf} \mathbf{F}_{II}\Big)$$
$$= \omega\Big(\sum_{S_0, T_0 \in \mathcal{I}^n_d} \det(E_{S(i)T(j)} + (2d - j - k/2)\delta_{S(i),T(j)})_{1 \leq i,j \leq 2d}\Big).$$

On the right-hand side above, $S \in \mathcal{I}^{2n}_{2d}$ is defined using $S_0 \in \mathcal{I}^n_d$ by $S(i) = S_0(i)$, $S(d+i) = n + S_0(i)$ $(1 \leq i \leq d)$, and T is defined from T_0 similarly.

The expression on the right-hand side is the image under ω of a sum of $2d \times 2d$ minors with entries in $U(\mathfrak{m})$. There are $\lfloor k/2 \rfloor$ generators of $S(\mathfrak{p})^K$ as (4.1), however the equation above is trivial when $n < d \leq \lfloor k/2 \rfloor$ since the right-hand side becomes an empty sum.

For proving the theorem, we use the following lemma to compute Pfaffians:

Lemma 4.2 (Ishikawa-Wakayama [IW00]). *Let R be a commutative ring and $d \leq n$. For $A, B \in \operatorname{Mat}(n, 2d; R)$, $X \in \operatorname{Sym}(n; R)$, define $P = {}^tAXB - {}^tBXA \in \operatorname{Alt}(2d; R)$. We then have*

$$\operatorname{Pf}(P) = \sum_{S \in \mathcal{I}^{2n}_{2d}} \operatorname{Pf}\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}_{SS} \det\begin{pmatrix} A \\ B \end{pmatrix}_{S,\bullet}.$$

In particular, when $X = I_n$ we have

$$\operatorname{Pf}({}^tAB - {}^tBA) = (-1)^{d(d-1)/2} \sum_{S_0 \in \mathcal{I}^n_d} \det\begin{pmatrix} A \\ B \end{pmatrix}_{S,\bullet},$$

where $S \in \mathcal{I}^{2n}_{2d}$ is defined using S_0 by $S(i) = S_0(i), S(d+i) = n + S_0(i)$ $(1 \leq i \leq d)$.

Proof. The first formula is due to Ishikawa-Wakayama [IW00, Corollary 2.1]. For the second formula we use two facts. The condition $\mathrm{Pf}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}_{SS} \neq 0$ implies the condition $S(d+i) = n + S(i)$ $(1 \leq i \leq d)$, and we have the formula $\mathrm{Pf}\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} = (-1)^{d(d-1)/2}$. These facts give the second formula. □

Proof of Theorem 4.1. Similarly to Case **R** and Case **C**, define matrices by

$$X = (x_{si})_{1\leq s\leq n, 1\leq i\leq k}, \qquad \overline{X} = (x_{\bar{s}i})_{1\leq s\leq n, 1\leq i\leq k},$$
$$\partial = (\partial_{si})_{1\leq s\leq n, 1\leq i\leq k}, \qquad \overline{\partial} = (\partial_{\bar{s}i})_{1\leq s\leq n, 1\leq i\leq k},$$

and we have the equations of matrices

$$\omega(\mathbf{G}_{II}) = \sqrt{-1}({}^t(X_{\bullet,I})\overline{X}_{\bullet,I} - {}^t(\overline{X}_{\bullet,I})X_{\bullet,I}),$$
$$\omega(\mathbf{F}_{II}) = -\sqrt{-1}({}^t(\partial_{\bullet,I})\overline{\partial}_{\bullet,I} - {}^t(\overline{\partial})_{\bullet,I}\partial_{\bullet,I}),$$

for $I \in \mathcal{I}^k_{2d}$. Using these equations we can prove the theorem as follows:

$$\begin{aligned}
&\sum_{I\in\mathcal{I}^k_{2d}} \omega(\mathrm{Pf}\,\mathbf{G}_{II}\cdot\mathrm{Pf}\,\mathbf{F}_{II}) \\
&= \sum_{I\in\mathcal{I}^k_{2d}} \mathrm{Pf}({}^t(X_{\bullet,I})\overline{X}_{\bullet,I} - {}^t(\overline{X}_{\bullet,I})X_{\bullet,I})\cdot\mathrm{Pf}({}^t(\partial_{\bullet,I})\overline{\partial}_{\bullet,I} - {}^t(\overline{\partial}_{\bullet,I})\partial_{\bullet,I}) \\
&= \sum_{I}\sum_{S_0\in\mathcal{I}^n_d}(-1)^{d(d-1)/2}\det\begin{pmatrix} X_{\bullet,I} \\ \overline{X}_{\bullet,I}\end{pmatrix}_{S,\bullet}\cdot\sum_{T_0\in\mathcal{I}^n_d}(-1)^{d(d-1)/2}\det\begin{pmatrix} \partial_{\bullet,I} \\ \overline{\partial}_{\bullet,I}\end{pmatrix}_{T,\bullet} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 4.2)} \\
&= \sum_{I,S_0,T_0}\det\begin{pmatrix} X \\ \overline{X}\end{pmatrix}_{SI}\det\begin{pmatrix} \partial \\ \overline{\partial}\end{pmatrix}_{TI} \\
&= \sum_{S_0,T_0\in\mathcal{I}^n_d}\omega\Bigl(\det(E_{S(i)T(j)} + (2d-j-k/2)\delta_{S(i),T(j)})_{1\leq i,j\leq 2d}\Bigr). \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.6)}
\end{aligned}$$

□

4.3 Invariance of the Capelli elements

The Capelli element is

$$C^{\mathbf{H}}_d = \sum_{S_0,T_0\in\mathcal{I}^n_d}\det(E_{S(i)T(j)} + (2d-j-k/2)\delta_{S(i),T(j)})_{1\leq i,j\leq 2d},$$

for $d = 1, 2, \dots, n$, which appears on the right-hand side of the formula of Theorem 4.1. Note that $\omega(C^{\mathbf{H}}_d)$ is zero when $\lfloor k/2\rfloor < d \leq n$, while $C^{\mathbf{H}}_d \neq 0$ for $1 \leq d \leq n$. As in Case **R** and Case **C**, the Capelli element $C_d^{\mathbf{H}}$ is not a central element of $U(\mathfrak{m})$, but an H-invariant element. We omit the proof of the following proposition.

Proposition 4.3. *The Capelli element is H-invariant, that is, $C_d^{\mathbf{H}} \in U(\mathfrak{m})^H$, where $H = Sp_{2n}(\mathbf{C})$.* □

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